Graph theory in mathematical olympiad and competitions

 Customer reviews //doi.org/10.1142/9789814271134_0001 //doi.org/10.1142/9789814271134_0003 //doi.org/10.1142/9789814271134_0007 //doi.org/10.1142/9789814271134_0009 2 The book is the first systematic research completely devoted to a comprehensive study of virtual knots and classical knots as its integral part. The book is self-contained and contains up-to-date exposition of the key aspects of virtual (and classical) knot theory. Virtual knots were discovered by Louis Kauffman in 1996. When virtual knot theory arose, it became clear that classical knot theory was a small integral part of a larger theory, and studying properties of virtual knots helped one understand better some aspects of classical knot theory and encouraged the study of further problems. Virtual knot theory finds its applications in classical knot theory. Virtual knot theory occupies an intermediate position between the theory of knots in arbitrary three-manifold and classical knot theory. In this book we present the latest achievements in virtual knot theory including Khovanov homology theory and parity theory due to V O Manturov and graph-link theory due to both authors. By means of parity, one can construct functorial mappings from knots to knots, filtrations on the space of knots, refine many invariants and prove minimality of many series of knot diagrams. Graph-links can be treated as "diagramless knot theory": such "links" have crossings, but they do not have arcs connecting these crossings. It turns out, however, that to graph-links one can extend many methods of classical and virtual knot theories, in particular, the Khovanov homology and the parity theory. Sample Chapter(s) Chapter 1: Basic Definitions and Notions (274 KB) Contents: Basic Definitions and Notions Virtual Knots and Three-Dimensional Topology Quandles (Distributive Groupoids) in Virtual Knot Theory The Jones–Kauffman Polynomial: Atoms Khovanov Homology Virtual Braids Vassiliev's Invariants and Framed Graphs Parity in Knot Theory: Free-Knots: Cobordisms Theory of Graph-Links Readership: Graduate students and researchers in combinatorics and graph theory and knot theory. "We hope that the reader of this review is motivated to delve into the adventure presented by this remarkable book." Journal of Knot Theory and Its Ramifications "The book is highly recommended to undergraduates, graduates, professionals and amateur mathematicians, because it goes from the basics to the frontiers of research." The European Mathematical Society "Indispensable for all devotees of combinatorics, graph theory, and (above all) knot theory." "The theory of virtual knots and related objects has expanded hugely since the first paper on the topic in 1999. This is, as far as I am aware, the first book on this topic and fulfills a definite need." Sample Chapter(s) Chapter 1: Basic Definitions and Notions (274 KB) Flipkart Internet Private Limited, Buildings Alyssa, Begonia & Clove Embassy Tech Village, Outer Ring Road, Devarabeesanahalli Village, Bengaluru, 560103, Karnataka, India CIN : U51109KA2012PTC066107 Telephone: 1800 202 9898 Thank you for your participation! VDOC.PUB Authors: Liu , Ruifang;Zhongyi , Zheng PDF Download Embed This document was uploaded by our user. The uploader already confirmed that they had the permission to publish it. If you are author/publisher or own the copyright of this documents, please report to us by using this DMCA report form. Report DMCA Graph Theory Mathematical Olympiad Series ISSN: 1793-8570 Series Editors: Lee Peng Yee (Nanyang Technological University, Singapore) Xiong Bin (East China Normal University, China) Published Vol. 1 A First Step to Mathematical Olympiad Problems by Derek Holton (University of Otago, New Zealand) Vol. 2 Problems of Number Theory in Mathematical Competitions by Yu Hong-Bing (Suzhou University, China) translated by Lin Lei (East China Normal University, China) Vol. 3 Graph Theory by Xiong Bin (East China Normal University, China) & Zheng Zhongyi (High School Attached to Fudan University, China) translated by Liu Ruifang, Zhai Mingqing & Lin Yuanqing (East China Normal University, China) Xiong Bin East China Normal University, China Zheng Zhongyi High School Attached to Fudan University, China Vol. 31 Mathematical Olympiad Series Graph Theory translated by Liu Ruifang Zhai Mingqing Lin Yuanqing East China Normal University, China ~ East China Normal ~ University Press ,It World Scientific Published by East China Normal University Press 3663 North Zhongshan Road Shanghai 200062 China and World Scientific Publishing Co. Pte. Ltd. 5 Toh Tuck Link, Singapore 596224 USA office: 27 Warren Street, Suite 401-402, Hackensack, NJ 07601 UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE British Library Cataloguing-in-Publication Data A catalogue record for this book is available from the British Library. GRAPH THEORY Mathematical Olympiad Series — Vol. 3 Copyright © 2010 by East China Normal University Press and World Scientific Publishing Co. Pte. Ltd. All rights reserved. This book, or parts thereof, may not be reproduced in any form or by any means, electronic or mechanical, including photocopying, recording or any information storage and retrieval system now known or to be invented, without written permission from the Publisher. For photocopying of material in this volume, please pay a copying fee through the Copyright Clearance Center, Inc., 222 Rosewood Drive, Danvers, MA 01923, USA. In this case permission to photocopy is not required from the publisher. ISBN-13 978-981-4271-12-7 (pbk) ISBN-10 981-4271-12-8 (pbk) Printed in Singapore by B & Jo Enterprise Pte Ltd Editors XIONG Bin East China Normal University, China Lee Peng Yee Nanyang Technological University, Singapore Original Authors XIONG Bin East China Normal University, China ZHENG Zhongyi High School Affiliated to Fudan University, China ZHAI Mingqing East China Normal University, China LIN Yuanqing East China Normal University, China Copy Editors NI Ming East China Normal University Press, China ZHANG Ji World Scientific Publishing Co. , Singapore WONG Fook Sung Temasek Polytechnic, Singapore KONG Lingzhi East China Normal University Press, China v This page intentionally left blank Introduction In 1736, Euler founded Graph Theory by solving the Königsberg seven-bridge problem. It has been more than two hundred years till now. Graph Theory is the core content of Discrete Mathematics, and Discrete Mathematics is the theoretical basis of Computer Science and Network Information Science. This book vulgarly introduces an elementary way some basic knowledge and the primary methods in Graph Theory. Through some interesting mathematical problems and games the authors expand the knowledge of Middle School Students and improve their skills in analyzing problems and solving problems. vii This page intentionally left blank Contents Introduction Chapter 1 Definition of Graph 1 Chapter 2 Degree of a Vertex 13 Chapter 3 Turán's Theorem 24 Chapter 4 Tree 40 Chapter 5 Euler's Problem 51 Chapter 6 Hamilton's Problem 63 Chapter 7 Planar Graph 75 Chapter 8 Ramsey's Problem 84 Chapter 9 Tournament 101 Solutions 110 Index 145 ix This page intentionally left blank Chapter 1 Definition of Graph ~ - .. ;~ Graph Theory is a branch of mathematics on the study of graphs. The graph we consider here consists of a set of points together with lines joining certain pairs of these points. The graph represents a set that has binary relationship. In recent years, graph theory has experienced and has generated extensive applications in many fields . We often encounter the following phenomena or problems: In a group of people, some of them know each other, some of them do not. There are some cities. Some pairs of them are connected by airlines and others are not. There is a set of points in the plane. The distance between some of them is one unit, others are not. All the above phenomena or problems contain two aspects: one is object, such as people, football teams, cities, points and so on; and the other is a certain relationship between these objects, such as "knowing each other", "having a contest", "the distance between" and so on. In order to represent these objects and the relationships, we could use a point as an object, which is called a vertex. If any two objects have a relationship, then there is a line joining them, which is called an edge. Then we have constructed a graph. We call the figure a graph Δ • For instance, the three graphs G , 1 , a) The general definition of graphs: a graph is a triplet (V, E, ,p), where V and E are two disjoint sets, V is nonempty and,p is a mapping from V x V to E. The sets V, E , ,p are vertex set, edge set and incidence function, respectively . Graph Theory 2 G2 , G 3 in Fig. 1. 1 are isomorphic, which contain some vertices and edges joining them, representing some objects and the relationships between them. Fig. 1. 1 shows three graphs G 1, G 2 , G 3 , where vertices are represented by small circles. ~~A v, v, G, v, v] G, Fig. 1. 1 We can see that in the definition of graphs there are no requirement on the location of the vertices, the length and the curvature of the edges, and the fact whether the vertices and the edges are on the same plane or not. However, we do not allow an edge passing through the third vertex and also not let an edge intersect itself. In graph theory, if there is a bijection from the vertices of G to the vertices of G' such that the number of edges joining u and v equals the number of edges joining u' and v' then two graphs G and G' are isomorphic and considered as the same graph. A graphG' = (V', E') is called a subgraph of a graphG = (V, E) if V' ⊆ V, E' ⊆ E, that is, all the vertices of G' are the vertices of G and the edges of G' are the edges of G. For instance, the graphs G 1 , G 2 in Fig. 1 . 2 are the subgraphs of G, G G, Fig. 1 2 G, Definition of Graph 3 If there is an edge joining v i and v j in graph G, then v i and v j are adjacent. Otherwise, they are nonadjacent. If the vertex v is an end of the edge e, then v is incident to e. In Fig. 1.3, adjacent, but vertex V1 V2 and V4 are not. The V1 and V2 are v, is incident to the edge e 4 • We called the edge is loop if there is v, e, an edge joining the vertex itself. For instance , the edge e 1 is a loop in Fig. 1.3 e, e, Fig. 1. 3 loop. Two or more edges that have the same pair of ends are called parallel edges. A graph is simple if it has no loops or parallel edges. The graphs G 2 , G 3 in Fig. 1. 1 are simple, whereas the graph in Fig . 1.3 is e1 , e2 not. In a simple graph, the edge joining v i and v j is denoted by (v i' V j). Certainly, (v i' V j) and (v j ' Vi) are considered as the same edge. A complete graph is a simple graph in which any two vertices are adjacent. We denote the complete graph with n vertices by K n. The graphs K 3 , K 4' K 5 in Fig. 1.4 are all complete graphs. The number of edges of the complete graph K" is G) = n (n - 1) . K, Fig. 1. 4 A graph is finite if both the number of the vertices IV I (IV I is also said to be the order of G) and the number of edges IE I are finite. If IV I or IE I is infinite, In this chapter, unless specified, all graphs under discussion should be taken to be finite simple graphs. Graph Theory 4 These fundamental concepts mentioned above help us to consider and solve some questions . Example 1 There are 605 people in a party. Suppose that each of them shakes hands with at least one person. Prove that there must be someone who shakes hands with at least two persons. Proof We denote the 605 people by 605 vertices V605 ' VI' V2' ... , If any two of them shake hands, then there is an edge joining the corresponding two vertices. Hence, we get a graph G. The degree of each vertex of G is at least one. If the conclusion is not true, then each of them just shakes hands with at least one person . Thus we have each of them just shakes hands with one person. It implies that the graph G consists of some disjoint edges. This is impossible since G has 2n (even) vertices. It contradicts the fact that the number of vertices of G is 605 (odd). Therefore the proof is complete. Fig. 1. 5 Example 2 Is it possible to place the numbers 1 to 9 into the grid in Fig. 1. 6 so that the sum of any two adjacent numbers is prime? Fig. 1. 7 by moving the knights several times? (In the figures , W stands for white knight, and B stands for black knight . knight should be moved by following the international chess regulation) Solution As Fig. 1. 8 shows, the nine squares are numbered and each of them is represented by a vertex in the plane. If the knight can be moved from one square to another square, then there is an edge joining the two corresponding vertices, as Fig. 1. 9 shows. ~~~ Fig. 1.6 Fig. 1.7 1 4 7 2 5 8 3 6 9 Fig. 1. 8 Definition of Graph '0 0 1 W B 9 3 5 8 3 Fig. 1. 9 0 I W . B 9 B 4 5 5 . 8 4 2 W 9 B 3 Fig. 1 10 4 Fig. 1.11 Thus the beginning state in Fig. 1. 6 and the state in Fig. 1. 7 are represented by the graphs as in Fig. 1. 10 , Fig. 1. 11, respectively. Obviously, the order of the knight on the circle cannot be changed from the state that two white knight are followed by two white knight into the state that white knight and black knight are interlaced. So it is impossible to change the states as required. Example 3 There are n people A1 , A 2 , • • • , A n taking part in a mathematics contest. Suppose that among them some people know each other and any two people who do not know each other would have common acquaintance . Suppose that A1 and A 2 know each other, but do not have common acquaintance. Prove that the acquaintances of Al are as many as those of A 2 • Proof Denote the n people Al, A 2 , ... An by n vertices V1 , ... V n. If two people know each other, then there is an edge joining the two corresponding vertices. Then we get a simple graph G. V2' The vertices of G satisfy that any two nonadjacent vertices have a common neighbor. We shall prove two adjacent vertices V1 and V2 have the same number of neighbors. The set of neighbors of the vertex set of neighbors of the vertex N (V2). If there is a vertex V2 Vi is denoted by N (V1) and the is denoted by n In N (V2) . Otherwise Vi and V2 have the common acquaintance A1 . V1# • V2 • V1 and V 1 So V shows For have a common neighbor (V2) Vi ' Vi contains n by V (V1 V') , v, VI' (V~ ' NCfJ ,) ,\" ~.. ,''' ' ' ~.-~' ' Vj and Vi ... Fig. 1. 12 which Graph Theory 6 distinct from V2' both of them adjacent to a vertex v j in N (V2)' which is distinct from V1 . Otherwise, two nonadjacent vertices V1 ' V have three common neighbors V2' Vi ' V k . Therefore , each vertex V2' must have a neighbor v I in N (V1)' which is distinct from v j . So the number of vertices in N (VI) is not greater than that of N (V2) . Similarly the number of vertices in N (V2) is not greater than that of N (Vl). Thus they are equal. Example 4 Nine mathematicians meet at an international mathematics conference. For any three persons , at least two of them can have a talk in the same language . If each mathematician can speak at most three languages, prove that at least three mathematicians can have a talk in the same language. (USAMO 1978) Denote the 9 mathematicians by 9 vertices VI' V2' ... , If two of them can have a talk in the ith language, then there is an Proof V9 edge joining the corresponding vertices and color them with the i th color. Then we get a simple graph with 9 vertices and edges colored. Every three vertices have at least one edge joining them and the edges incident to a vertex are colored in at most three different colors. We need to prove that there exist two of the three edges colored with the same color. (This triangle is called monochromatic triangle.) Have the i th color, then the vertices v j , V k are adjacent and edge (v j , V k) has the i th color. Thus for vertex VI , there are two cases : (1) The vertex V I is adjacent to V 2 ' . . . , V 9. By the pigeonhole principle, at least two edges, without loss of generality, denoted by the edges (v i ' V j) , (v i ' V k) have the same color. Thus triangle monochromatic . (VI' V2), (VI' V 3) ' D V I V2V 3 is a (2) The vertex VI is nonadjacent to at least one of V 2' ... , V 9 . Without loss of generality, we suppose that VI is nonadjacent to V 2 . For every three vertices there is at least one edge joining them, so there are at least seven edges from vertices V3 ' V4' ... , V9 to the Definition of Graph 'Vertex VI or V 2' From that we know at least four VI' V2 vertices of V3' V4' ... , V9 are adjacent with vertex VI or V2' Without loss of generality, we suppose that V3, V4' V5, V6 are adjacent with vertex VI or as it is shown in Fig. 1. 13. Thus we know there must exist two v 6 edges of (VI' V3) (VI' V4), (VI' V5) (VI' V6) which have the same color. Suppose (VI' V3), (VI' V4) have the same color, then DVIV3V4 is a monochromatic triangle . Remark If the number 9 in the question is replaced by 8, then the proposition is not true. Fig. 1.14 gives a counterexample. Denote the 8 vertices by VI' V2' . . . , V8 and 12 colors by 1, 2, ... , 12, and the edges are colored by the corresponding numbers as it is shown in the graph. ~, I~~ (V, I?~~~ ~ ~ /'1 v, V, 8 Fig. 1. 14 The following example is from the (Chinese National Mathematical Olympiad 1988). Suppose that among the persons by the vertices V I, V2 , ... , V n. Then there is an edge Ai A j having one of the vertices in the graph and each of them has ... Example 5 Prove that among 2n +1 ... A1 , A2 , ... , An... As a special case... Let a 1, a2, ... be ... We join... Example 6 ...

game of go, Chinese checkers. It is known that everyone must play Chinese chess with n persons, the game of go with n persons and Chinese checkers with n persons. Prove that among the 3n + 1 persons, there must be three persons so that there are Chinese chess player, go player, and Graph Theory 96 Chinese checkers player among the three. Proof We denote 3n + 1 persons by 3n + 1 vertices. If two persons play Chinese chess, the corresponding edge of them is colored red. If two persons play the game of go, the corresponding edge of them is colored blue. If two persons play Chinese checkers, the corresponding edge of them is colored black. Then we get a 3-color complete graph K 3n+1. What we must prove is that in this 3-color complete graph K 3n + 1 , there must be a heterochromous triangle. If two edges adjacent to one vertex are not monochromatic, we call the angle of the two edges heterochromous angle. A triangle is heterochromous if and only if its three angles are heterochromous angles. Every vertex is adjacent to 3n edges during which there are n red edges, n blue edges, n black edges, respectively. Therefore, the number of the heterochromous angles induced by one vertex is (~) n 2 The 3-color complete graph K 3n+1 contains 3n 2 On + 1) heterochromous angles at all. On the other hand, complete graph 3n 2. 3n + K 3n+1 contain (3 1) 21 On + 1) On -1) triangles. We can regard = n these triangles as holes and heterochromous angles as pigeons. Since 3n 2 On + 1) >n On + 1) On - 1), the number of heterochromous angles in the 3-color complete graph K 3,,+1 is twice more than the number of triangles. By the Pigeonhole Principle, there must be a triangle with three heterochromous angles. We call the triangle a heterochromous triangle. We often find the similar problems about the Ramsey problem in mathematics contest. We give some more examples to finish this chapter. Example 7 There are 100 guests in a hall. Everyone of them knows at least 67 persons. Prove that among these guests you can find 4 persons any two of them know each other. (Polish Mathematical Competition in 1966) We denote the guests by 100 vertices A1 , A 2 , ••• , Awo. Join every two vertices and color it in red or blue. The edge joining A; and Aj is Proof Ramsey's Problem 97 colored in red if and only if A. and Aj know each other. We use the language of graph theory to re-phrase this problem: In a red-blue two color complete graph K tOO' if the number of red edges going out of every vertex is at least 67, then K IOU contains a red complete subgraph K 4 • Take one vertex AI. The number of red edges induced by it is no less than 67, so there must exist a red edge A I A 2 • Since the number of red edges induced by A2 is no less than 67, the number of blue edges induced by Al and A2 is at most 32 X 2 = 64. They involve 66 vertices and there must exist one vertex, for example A3 so that AIA3 and A2A 3 are red edges. The number of blue edges induced by AI, A 2 , A3 is at most 32 X 3 = 96 and these blue edges involve 99 vertices. There must exist one vertex A4 so that AIA4' A 2A 4 , A3A4 are red edges. So the complete subgraph K4 with vertices AI, A 2 , A 3 , A4 is red. Example 8 We use pentagons AlA2A3A4AS and BlB2B3B4BS as the top and bottom faces of a prism. Every edge and every line segment AiBj' where i, j = 1, 2, ... , 5, are colored in red or blue. Every triangle which uses a vertex of the prism as its vertex and a line segment which has been colored as its edge is not a monochromatic triangle. Prove that the ten edges in the top and bottom faces are colored the same color. (The 21th IMO) Proof First we prove that the five edges on the top face are colored the same color. Otherwise, there are at least two edges in the pentagon which are not monochromatic. So there are two adjacent edges such as A I A 2 , AlAs which are not monochromatic. Without loss of generality, we suppose that AlA2 is red and AlAs is blue. Among the edges joining AI and B l , B 2 , B 3 , B 4 , Bs there are at least 3 monochromatic edges. Let AIB i . AIBj' AIBk be red edges (i, j, k are distinct). Since 6A I BiB j is not monochromatic, BiB j is a blue edge. Similarly, A2Bi is also a blue edge. We can also know that A2B j is a red edge. Then 6AlA2Bj is a red triangle. It is a contradiction. Similarly, we can prove that the five edges on the bottom face are also monochromatic. If the edges in the top and bottom faces are not monochromatic. 98 Graph Theory We suppose that AtA2A 3A 4A s is red and BtB2B 3B4B s is blue. Without loss of generality, let AtB I be a blue edge. By the assumption that every triangle is not monochromatic, we can know that Al Bs and AIB2 are all red edges. So AzB 2 and AsB s are blue edges. Similarly, A sB I' A sB4' AzBI , A 2 B 3, . . . are all red edges and A 3B 3' A4B4 are blue edges. So A 4BI and A4BZ are blue edges and we can get a blue triangle D A4Bt B 2 • It is a contradiction. So the ten edges in the top and bottom faces are monochromatic. Example 9 10 districts. There are two international airlines X and Y serving For any two districts, there is only one company providing a direct flight (to and fro). Prove that there must be a company which can provide two tour routes so that the two routes do not pass through the same districts and each route passes through an odd number of districts. We denote the 10 districts by 10 vertices Ut , U 2 ' • . . , U IO. If the flight between U i and U j is provided by X, then we join U i and U j by a red edge (a solid line): If the flight between U i and U j is provided by Y, then we join U i and U j by a blue edge (a dotted line). Then we can get a 2-color complete graph K 10. In order to prove the conclusion, it suffices to prove that there must be two monochromatic Proof triangles or polygons having no common edge and an odd number of edges in K 10 • The 2-color complete graph K 10 contains a monochromatic triangle. Let D USU9U1I1 be a monochromatic triangle. By Example 1, we can know that the triangles constructed by the vertices Ut, U2' • . • , U 7 must contain a monochromatic triangle . Let D U S U 6 U 7 be a monochromatic triangle. If the color of DU s U 6 U 7 is the same as that of the conclusion holds. Then let be blue. D USU9U tu, D USU9UIO D U SU 6 U 7 be red and The number of edges joining the vertex sets {u s , U 6' U 7} and {u s , U 9' U IO} is 3 x 3 = 9. By the Pigeonhole Principle, there must be five monochromatic edges . Let them be red edges . The five edges are induced by {us, U9 ' U IO} , so there must exist a vertex which is Ramsey's Problem adjacent to two red edges which are U8U6, there must also be another red triangle 99 U g U7' As Fig. 8. 8 shows, DU6U7US. Consider the 2-color complete graph Ks whose vertices are U2' U3' U4' us. If the Ks contains a monochromatic triangle, whatever color the triangle is, together with the red triangle D U S U9 U 10' K 10 contains two monochromatic triangles with common edge and the same color. Otherwise, the 2-color complete graph K s contains no monochromatic triangle. It is easy to know that Ks contains two monochromatic pentagons which are one red and one blue. We complete the proof. Remark Ul' D U6 U 7 Us ".~" U9 .~~ - - - - - - - -.. :~.UIO Fig. 8. 8 If we replace 10 districts by 9 districts, the conclusion is false. An example is given as follows. We divide 9 districts into 3 groups, i.e. {Ul' Uu U 3 ' Up us} = A, {U6' U7' us} =B, {Uy} = C. The flights among the five districts in A are provided by X. The flights among the three districts in B are provided by Y. The flights between A and B, U 9 and A are provided by Y: The flights between A are provided by X . Ug Exercise 8 1 In the space, there are six points. Join every two of them and color the lines in red or blue. Prove that there must be two monochromatic triangles which contain no common point. 2 In the space, there are eight points. Join every two of them and color the lines in two colors. Prove that there must exist three monochromatic lines which contain no common point. 3 In the space, there are six points. Any three points are not the vertices of an equilateral triangle. Prove that among these triangles, there is one triangle whose shortest side is also the longest line of another triangle. Graph Theory 100 4 Join nine distinct points on a circle to get 36 lines and color them in red or blue. Suppose any triangle with three vertices coming from the nine points contains a red line. Prove that there are four points and any edge joining two of them is red. S Prove that among any 19 persons, there must be 3 persons who know each other or 6 persons who do not know each other. 6 Prove that among any 18 persons, there must be 4 persons who know each other or do not know each other. 7 We divide the natural numbers 1, 2, . . . , N into n groups. When N is large enough, there must be a group which contains x, y and their difference 1x - y I. (Schur Theorem) 8 There are 1978 members in an international corporation. They come from 6 countries. We label them as 1, 2, . . . , 1978. Prove that there must be at least one member whose number is twice the number of his one fellow-country or the sum of two fellow-countrymen. 9 Prove that in a 2-color complete graph K 7 ' there must be two monochromatic triangles with no common edges. 10 In the space, there are six lines . Among them every three lines do not lie on a plane. Prove that there must be three lines satisfying one of the following three conditions: (i) Any two of them do not lie on a plane . (ii) Any two of them is parallel to each other. (iii) They meet at one point. 11 Find the minimum positive integer n so that any given n irrational numbers always contain three irrational numbers among which the sum of any two is also an irrational number. 12 Find the minimum positive integer n so that when the K is colored by two colors arbitrarily, there must be two monochromatic TI triangles which are colored by one color but contain no common edges. 13 In a football league, there are 20 football teams. In the first round , they are divided into 10 matches. In the second round, they are also divided into 10 matches. (Notice that the opponent of every team in different rounds can be the same.) Prove that before the third round, you can find 10 teams which have not played with each other. Chap ter 9 Tournament In Chapter 1, we have said that the graph is a tool to describe the special relationship of some objects. The graph in the above chapter is an undirected graph. The relationship they describe is a symmetrical relationship. In daily life, many relationships are not symmetrical such as the relationship of knowing each other. When X knows y, it does not mean that Y knows X. So is the relationship of winning or losing in a match. So we can have a new definition of directed graph. We call a graph directed graph if we assign to every edge of the graph a direction. We call the edge of a directed graph an arc. If there is an arc joining the vertices v i and v j and the arrow of the arc points from v i to V j' we denote it by (v i ' V j) and call v i the starting point and call v j the end point. Generally, we denote the directed graph by D = (V , U). Here we denote the vertex set of D by V and the arc set of D by U. Fig. 9.1 shows us a directed graph. The vertex set is and the arc set is U = {(v ,' V2)' (V 2' V3) ' (vs , V2) ' (vo V2), (V4' V6), (VS , V6)' (VS , V 4), (V3 ' v s), (V4 ' vs) } . The directed graph in this chapter is also a simple directed graph , which is a graph without loops (an arc which starts and ends at the same point) and without multi-arcs (there are more Fig. 9. 1 Graph Theory 102 than one arc joining v i and v j) • We say that v i and v j are adjacent if there is an arc (v i ' V j) or (v)' Vi) in the arc set of the directed graph G. Otherwise, we say that v i and v j are not adjacent. We call the number of the arcs whose starting points are v i an outdegree of Vi' which is denoted by d +(v i). We call the number of the arcs whose end points are v i an indegree of v" which is denoted by d- (v i). We call a directed graph tournament graph if the graph contains n vertices and there is only an arc joining every two vertices. We denote the directed graph by K" . Let V1' V2' ... , V n be the vertices of tournament Theorem 1 Kn. Then d +(V1) +d+(V2) d+(v n) =d - (V1) +d - (V 2) +"'+d-(v n) = Proof + ... + 1 2n(n -1). Since every arc of K" induces one in degree and one outdegree and there is only one arc joining every two vertices, the sum of indegree of every vertices in K n is the same as the sum of outdegree of every vertices. d +(V1) +d+(V2) + ... +d +(v n) =d - (V1) +d-(V2) + "'+d-(v,,) Example 1 = 1 2n(n -1). n players P 1 , P 2 , ••• , P" (n > 1) take part in a round robin. Every player plays only one game with any of other n -1 players. Suppose that there is no tie in the result and we denote the number of win and lose of P r by w r and l TO respectively. Prove that wf +w~ + ···+w ;, =n +l~ + ··· + l~. (The 26th American Putnam Mathematical Competition) Solution Draw a tournament K". We denote the person P r by Tournament the vertex v r ' 103 If Pi defeats P j ' we join v i and v j to get an arc (v i , j) . So W rand lr are the indegree and outdegree of v r respectively. By Theorem 1, V Wl+W2 + ···+Wn =ll+ l 2+···+ l n. Note that Wi WT + w~ = (WT +li = n + ... +w~ - in +(W~ - ~ -1 C1 (iT +l~ i ~ n), + ... + l~) - lD +··· + (w~ - l~) = (WI + ll)(wl -ll) + (W 2 +l2)(W 2 - l2) +··· + (w n + In)(w n - In) =(n - 1)[(wl +W2+ ···+Wn) - (l1 +l2+··· + ln)]J =0. So wf +W ~ + ... +W~ In a directed graph D = = (v, u), If + n + ... + l~. there exists a sequence of distinct arcs U I ' U 2' . • . , Un . If the starting point of U i is v i and the end point of Ui is Vi+1 Ci = 1, 2, ... , n). We call n the length of the directed path. VI is the starting point of the path and v n + I is the end point. If v I = V n+ 1 , we call the path a circuit. Example 2 The MO space city consists of 99 space stations. Any two stations are connected by a channel. Among these channels there are 99 two-way channels and others are one-way channels. If four space stations can be arrived at from one to another, we call the set of four space stations a connected four-station group. Design a scheme for the space city so that we get the maximum number of connected four-station groups. (Find the exact number and prove your conclusion.) (The 14th China Mathematical Olympiad) Solution We call an four-station group a bad four- station group. A bad four-station group has three possible situations: (1) Station A has three channels AB , AC, AD which all leave A. (2) Station A has three channels which all arrive at A. (3) Stations A and B, stations C and D have two-way channels but the channels AC, AD all leave A, and BC, BD all leave B. We denote all the bad four-station groups in (1) by S and others Graph Theory 104 by T. Let us calculate IS I. Since the space city contains C;) - 99 = 99 x 48 one-way channels . We denote the number of channels leaving the i - th station by S i ' So 99 ~Si 99 X 48. = i= 1 Now the number of bad four-station group in (1) which contains (~i). three channels induced by A is IS I= 99 ~ 1~ 1 (So S.)~ 99 x (48) . I 3 3 The above inequality holds because x ~ is (~) = ~ x(x - D(x - 2) for 3 is a convex function . Since the number of all four-station groups (9:), so the number of connected four-station groups is no more than Then we give an example so that the number of the connected four-station groups is (9:) - 99(~8) . Let the number of the channels from and to every station Ai be both 48. Every station has two twoway channels and there are only type S groups of bad four-stations and no type T groups of bad four-stations. We put 99 stations on the vertices of an inscribed polygon with 99 sides and assume the longest diagonals of the regular polygon with sides all two-way channels. So every vertex is adjacent to two two-way channels. For station A i , there are one-way channels leaving Ai and joining the next 48 stations in the clockwise direction, and one-way 99 Tournament 105 channels arriving at Ai and joining the next 48 stations in the counterclockwise stations. Then we will prove there are only type S groups of bad four-station groups and no T groups of bad four-station groups. Suppose {A, B, C, D} is a four-station group. (i) If there are two two-way channels in the four-station group, clearly they are connected. Cii) If there are only one two-way channel AC in the four-station group, each of Band D forms a cycle with A, C. Of course, they are connected. (iii) So if the bad four-station group contains no two-way channel, it can only be one of (1) and (2). If it is (2), without loss of generality, we suppose that the 3 channels of A all arrive at A and B, C, D all come from the next 48 stations of A in the clockwise direction . Let D be the farthest station from A. So AD, BD, CD all leave from D, which means that all the bad four-station groups are of the type S group. In summary, there are at most C:) - 99(~8) connected four- station groups. Theorem 2 There exists a vertex in a tournament so that there is a path from it to any other vertices. The maximum length of the paths is 2. Proof Suppose that the vertex with the maximum outdegree of a tournament Kn is v\. We denote the end point set of the arcs whose starting point is V1 by N+ (v 1). If the conclusion is false, then there must be a vertex V 2 (v 2 eft v \ ), where v 2 tl: N + (v \ ). For every vertex u E N+ (v \) , there is one arc (V2' u) from V2 to u together with the arc (V2' v\) . So d+ (v 2) "?:- d+ (v \) + 1, which contradicts the fact that degree of v\ is maximum. The proof is complete. Every athlete who takes part in the single round robin must play one game with any other athlete and there is no tie. Prove that among these players, you can find such athletes that the persons Example 3 who were defeated by him and the persons who were defeated by the Graph Theory 106 person he defeats can contain all other athletes. (Hungarian mathematic contest) Use tournament on the round robin, which is Theorem 2. We omit the proof. n (n ~ Example 4 3) athletes take part in a single round robin and use the result to find good athletes. The requirement that A is selected to be a good athlete is that for any other athlete B, either A defeats B or there exists C so that C defeats B and A defeats B. If only one athlete meets the above requirement, show that he defeats any other athletes. Solution We denote n athletes by n vertices. If draw an arc from v i to v j and get a tournament K generality, we suppose that the outdegree of according to Theorem 2, that VI VI VI v defeats i v j , we Without loss of n. is maximum in the K" , is a good athlete. What we will prove is can arrive at any other vertex by a path whose length is 1, which means that the indegree of VI is d- (v I) = O. Suppose that the proposition is false. We denote the set of arcs with starting point VI by N- (v I) Consider the K r consisting of VI' V i 2 ' outdegree of v i l is maximum in K from v il to each of v i2' •.• , V ir at other vertices except {v i 1 at vertices except VilI' V i 2 ' = r. ••• , V i2' Vir. ••• , Vir}' 1. By Theorem 2, the length of path ... , Vi' Vi}' . . . , V ir r ~ We suppose that the is no more than 2. Since , Vi' 2 {v iI' r 1 then VI Vi 1 can arrive can arrive through the paths whose length is no more than 2. Therefore, in the tournament K r , Vi I can arrive at any other vertex through a path whose length is no more than 2. Hence VilI is also a good athlete, which contradicts the fact that the only good athlete. So N- (V I) = completed the proof. Remark 0 VI is or d- (v I) = O. We have This problem gives a property of the tournament IL: If the vertex of Kn with the maximum outdegree is unique, then the outdegree of this vertex is n - 1. Theorem 3 Tournament K n contains a Hamiltonian path whose Tournament 107 length is n - 1. Proof Apply induction on the number of vertices n. When n = 2, clearly the proposition is true. ~ Suppose the proposition is true for n k v k. When n = k + 1, from + 1 vertices we take a vertex v . Remove v and the arcs adjacent to from R u 1 . By induction, R k+ 1 - v contains a Hamiltonian path VI' If there is an arc (v k' v), then Hamiltonian path. If there is an arc is also a Hamiltonian path. (v , VI' VI)' V2' then v , V k' VI' V2' V IS a ... , V k V Otherwise there exist arc and (VI' v). Then there must be an i C1 ~ i ~ k - 1) so that the arcs (v;, v), (v, Vi + I) (v, .•. , V k) VI V2 v) both exist. Now VI' . . . , v;, v, Fig.9.2 v; + l ' . . . , V k is a Hamiltonian path as Fig . 9. 2 shows us. Example 5 In a match of Chinese chess, every two players should playa game. Prove that we can label the players so that every player is not defeated by the player whose number follows immediately after his . Suppose there are n players. We denote n players by n Solution vertices VI' V2' arc from v i to ... , V j v". to get When (v i' v V j). i is not defeated by v j' we draw an Then we get a tournament K". By Theorem 3 , K" contains a Hamiltonian path, so we can label them according to the order of the path. Theorem 4 The tournament K" (n ;?-3) contains a circuit which is a triangle, if and only if there are two vertices v and v' satisfying Proof Let vertices v and v' satisfy d+ (v) d+ (v'). We will prove that K" contains a circuit which is a triangle. Without loss of generality, we assume that there is an arc (v , v') Graph Theory 108 and draw arcs from V'to everyone of d+ (v)). Then there must be Vj to v. Otherwise, d+ V j C1 (v) ~ k VI' ~ j ~ k) +1 > V2' ... , V k where (k = so that there is an arc from d+ (v') and v, v', V j form a triangle . We have proved the sufficient condition . If the outdegree of every vertex of K n is different, we can prove by induction that K n contains no triangular circuit. When n = 3, it is easy to see that the outdegree of a triangle is 0, 1, 2 and the triangle cannot form a circuit. Suppose the proposition is true for n = k. Consider the tournament K n +I ' If the outdegree of every vertex is different, they are 0, 1, 2, "', k in order. Supposed+(v ') = k. Remove the vertex v' and its adjacent arcs. By induction hypothesis, circuit. Clearly K k+1 Kk - v' contains no triangular contains no triangular circuit. We have proved the necessary condition. Exercise 9 1 Among n (n > 4) cities, every two cities have a path to join them. Prove that we can change these paths to one-way path so that it is possible to go from one city to another city through at most one other city. 2 If a tournament K trianglular circuit. 3 n contains a circuit, then K n contains a In a country, N cities are connected by air routes. For any route, the airplanes can fly only along one direction. An air route satisfies the condition f: Any plane which starts from one city cannot return to the same city. Prove that it is possible to design an airline system so that every two cities are connected by an air route and the system also satisfies condition f. 4 In a volleyball round robin, if team A defeats team B or team A defeats team C and team C defeats team B, we say that A is Tournament 109 superior to B and we also call the team superior to any other team the champion. According to this regulation, can two teams both win the championship? 5 n players take part in a match in which everyone plays with several other players. Suppose that there is no tie in a game . If the result that VI defeats V2' V2 defeats V 3 ' . . . , V k defeats VI does not appear. Prove that there must be a player who wins all games and another player who loses all games. 6 If among n persons v I ' V2' . . • , V n every two persons v i and v j have one ancestry v k. Everyone can be an ancestry of oneself. Prove that the n persons have a common ancestry. 7 A, B, C, D play table tennis and every two persons should play against each other. At last, A defeats D and A, B , C win the same number of games. How many games does C win? :;?o 3) players take part in a round robin. Every pair should playa game and there is no tie . There is no player who defeats all other players. Prove that among them there must be three persons A , 8 n (n B, C so that A defeats B, B defeats C and C defeats A. 9 There are 100 species of insects. Among every two of them there is one species who can eliminate another species. (But A eliminates B, B eliminates C, which does not mean that A eliminates C .) Prove that the 100 species of insects can be arranged in an order so that any species can perish another species next to it. Solutions 1. Consider a complete m-partite graph G' , the number of vertices of its m parts is nl , n 2' . . . , n i - 1, . . . , n j + 1 , . . . , n m Since • e CG /) 1 +2 (n - n J - 1)(n J + 1) = eCG) + (n i- n j) - 1 > eCG) . If G' is isomorphic to T m (n), we complete the proof. Otherwise, repeat the above step until we find a graph which is isomorphic to T m . n ' 6 Construct a bigraph G = CX, Y ; E) as follows. We denote every student from country X by a vertex in X and every student from country Y by a vertex in Y. If a student from X has danced with a student from y, then join the vertices corresponding to them. Suppose that the degree of x is the largest in X. Since d (x) < n, in y, there is a vertex y' which is not adjacent to x. Suppose that x ' in X is adjacent to y ' . Since except y', there are d (x ') - 1 vertices adjacent to x' and de x /) - 1 ~ de x) - 1 < de x), so there must be a vertex y which is adjacent to x but not adjacent to x ' . Then we get four vertices x , x ' , y, y' corresponding to four persons who satisfy the requirement. 7 Construct a graph G as follows. We denote 14 persons by 14 vertices . Two vertices are adjacent if and only if these two corresponding persons have not cooperated . By assumption, there are C24) = 91 pairs among 14 persons . Everyone has teamed up with the 14 X 5 other 5 persons, so there are - 2- = 35 pairs. Now they play 3 sets and have 6 new pairs . The number of edges of G is 91 - 35 - 6 but e2 (14) = = 50 , 49. By Tunln's theorem, G contains K 3 and the travelers Graph Theory 120 corresponding to three vertices can playa set with the new traveler. 8 SetG = (V, E). There ared(x){n -1 - d(x)} triple group {x, y, z}. They do not form a triangle in G or G, and x E V is the end of the only edge in G. Every triple group {x, y, z} which does not form a triangle in G or G contains one or two edges of G. Suppose that (x, y) is one edge of G and (x, z), (y, z) are two edges of G. In the sum ~d(x){n -1 -d(x)}, the triple group {x, y, z} has rEV been counted twice: one is about x, the other is about y. If (x, y), (y, z) are the edge of G, (x, z) is the edge of G, in above sum, the tuple group {x, y, z} has also been counted twice: one involves x, the other involves z. The sum of triangles in G and G is G)- ~ ~rEvd(x)(n = -1 - d(x» 1 24n(n -D(n - 5). 9 Suppose there is no (k + 1) -element set which we need. We will prove m < -'-(k'-'-----~1)'_(~n'_-----.:..:k'_'_)_+_'_____'_=k (k2 n) k - 1 . We denote all the red k -subsets by S and all the (k - 1) -subset by f3 . For any (k - 1) -subset B, we denote the number of red k -element subsets which contain B by a (B). For any A E S, A contains k (k 1)-element subsets. For any element x E X\A, x together with at most k - 1 of k (k - 1) -element subsets form a red k -element subset. (Otherwise, there exists (k + 1) -element subset and all its k -element subsets are red k -element subsets.) So ~ BcA, IBI= k-l So a(B)«n - k)(k-1)+k. Solutions m [Cn - k)Ck - 1) 121 + k] ~ ~ ~ aCB) A E SB C A. IB I ~ k - l = ~CaCB»2 B E {3 ~ m((;{3aCB») 2 ~ 1 - --Ckm)2. (k ~ 1) So ~ m ----, [Cn - k)(k - n + k](n) _ _ _ _ ~ _ _~k~-_1~ k2 Since [Cn - k)Ck - n + k](m n) > _____~~-_k---1k2 there must exist a Ck + 1) -element subset of X such that all k-element subsets are red k-element subsets. 10 Since C20) = 45, then a complete graph with 10 vertices contains 45 edges . The figure in the problem is obtained by removing 5 edges from the complete graph with 10 vertices. We call the 5 edges "Removed Edges" and denote 10 vertices by A l ' A 2' . . . , A 10 . Without loss of generality, let A 1 A2 be a "Removed Edge", then remove Al and its incident edges. The deduced graph with 9 vertices contains at most 4 "Removed Edges". Without loss of generality, let A2A 3 be a "Removed Edge". Then remove A2 and its incident edges. CIf there is no "Removed Edge", remove any vertex . The same in the later step.) The deduced graph with 8 vertices contains at most 3 "Removed Edges". Without loss of generality, let A3A4 be a "Removed Edge". Then remove A3 and its incident edges. The deduced graph with 7 vertices contains at most 2 "Removed Edges". Graph Theory 122 Without loss of generality, let A4AS be a "Removed Edge". Then remove A4 and its incident edges. The deduced graph with 6 vertices contains at most 1"Removed Edge". The deduced graph is a complete graph with 6 vertices or the graph which is obtained by removing one edge from the complete graph with 6 vertices. Anyway, the graph must contain a complete bipartite graph K 3 , 3 . We can generalize this problem: If a graph with n vertices and m edges contains no K can prove that m 11 X2' < n, n' C • n 2--;' , where C depends on r. We denote the positions of 18 police cars by 18 vertices ... , X IS . We Xl' Suppose By Theorem 3, I E I? (18) 2 - [182J 3 = 45. It means there are at least 45 pairs of cars which can communicate with each other. If the above condition of the graph does not hold, then there does not exist two vertices whose degrees are more than 5 and I E I~ ~ (1 X 17 + 4 X 17) < 43, a contradiction. 12 When n = 2, n 2 + 1 = 5. There are 5 line segments among 4 vertices, which form two triangles . Suppose that proposition is true for n = k. When n = k + 1, let us prove that there exists at least one triangle . Suppose that AB is a given line segment and denote the number of line segments from A and B to other 2k points by a and b. ? 2k + 1 , there exists a vertex C other than A and B so that AC and BC exist. Then there exists a triangle L:,.ABC. (2) If a + b ~2k, if we remove A and B, among the remaining 2k vertices, there exist at least k 2 + 1 line segments. By induction, there exists a triangle. (1) If a +b Suppose L:,.ABC is a triangle formed by these line segments. We Solutions 123 denote the number of line segments from A, Band C to other 2k - 1 points by a, (3 and y. (3) If a + (3 + y ~ 3k - 1, the total number of the triangles including one of AB, BC, CA as an edge is at least k . These k triangles together with L"ABC give k + 1 triangles. (4) If a + (3 + Y ~ 3k - 2, there is at least one number which is no more than 2k - 2 among the three numbers a +(3, (3 + y, y + a. Without loss of generality, a +(3 ~ 2k - 2. When we remove two vertices A, B, among the remaining 2k vertices there exist at least k 2 + 1 line segments. By induction, there exist at least k triangles. These k triangles together with L"ABC give k + 1 triangles. The proposition is true when n = k + 1. We complete the proof by induction. Exercise 4 1 The spanning tree of graph G contains two pendant vertices. Remove these two vertices and the graph is still connected. There are 9 2 X 9 81 vertices whose degrees are 4, so we should = remove at least [821] + 1 = 41 edges so that the degree of every vertex is less than 4 . We can remove at most 2 X 11 X 10 - 120 = 100 edges so that the graph is still connected. 3 The proposition is not true. Take K 3 and an isolated vertex (the vertex which is not adjacent to any other vertex) to construct a graph G. Then G contains 4 vertices and 3 edges, so it is not connected and clearly not a tree. 4 (1) Suppose that T has x pendant vertices. The number of vertices of the tree T is n = 3 + 1 + x, the edge number is e = n - 1 = 11 X 2: d (v,) + 3. = 3 X3 +2 Xl +1Xx = 11 + x, so 11 + x = 2(x + i= l 3), x = 5. (2) Fig. 4 shows us two trees satisfying the requirement but they are not isomorphic. Graph Theory 124 Fig. 4 5 Suppose T contains n vertices and e edges. Then n e = n - 1, k k ~ in i 2e = = 2n - 2 = 2~ n i - 2. i= 1 i= 1 So k nl ~(i - 2)n i +2. = i= 2 For r 6 ~ 3, by the above equality, we can obtain Among d l ' d 2' equal to 1. (Otherwise, ... , t di d n' there must be at least two which is ~ 2n - 1). We apply induction on the i =l number of vertices n . When n = 2, the proposition is true. Suppose that the conclusion is true when n = k . When n = k + 1, there exists a number 1 among d 1 , d 2 , ••• , d k , d k + 1 • Without loss of generality, let d k + l = 1. It is easy to know among the k + 1 numbers there exists a number which is no less than 2, denoted by d k. Consider the k numbersd 1 , d 2 , ••• ,dk-l , (d k - 1), d 1 + .. ·+dk-l + (d k - 1) = 2(k + 1) - 2 - 1 - 1 = 2k - 2. By induction, there exists tree T' whose vertices are VI' ••. , V k , k ~d(Vi) = d 1 + ... +d k - 1 + (d k - 1) = 2k - 2. i=1 In T', there is an edge which is from Vk to Vk + l ' We obtain a tree 125 Solutions T, then k+1 ~d(v;) = 2k -2 +1 + 1 = 2(k + 1) - 2. i= l T is what we need. 7 Construct a graph G, we use the ends of n line segments as the vertices of G and the line segments as the edges. By the condition, G is connected and contains no loop. So G is a tree where the length of its longest chain is 2. So G contains only one vertex which is not a pendant vertex. The vertex is the common vertex of n line segments. 8 Refer to Example 6 in this chapter. 9 Suppose the conclusion is not true, then there must exist a counterexample. Consider the counterexample in which the smallest. In this counterexample, 1E 1= 1V 1+ 4. 1E 1+1 A contradiction!) Then 1E 1> 1V I. 1is (Otherwise, we can remove more edges and still get a counterexample, where 1V 1is smaller. V 1 E 1 + The graph must contain a cycle. The length of the shortest cycle is at least 5. (Otherwise, the length of the shortest cycle is no more than 4, then we remove this cycle. We still have 1E 1;;: 1V I. There still exists a cycle. The cycle and the above cycle contain a common edge. A contradiction!) Furthermore, the degree of every vertex is at least 3. Otherwise, if the degree of a vertex is 2, remove this vertex and change the two edges adjacent to this vertex to one edge. We still have 1E 4 and 1E 1+ 1V 1gets 1= 1V 1 + smaller. A contradiction! If the degree of one vertex is 1, remove this vertex and its adjacent edges, we still have 1E 1= 1V 1+4 and 1E 1+1 V 1is smaller. A contradiction! If there exists an isolated vertex, remove it and 1 V 1 1E I > 1V 1+ 4 and 1E 1+ is smaller. A contradiction! Take a cycle Co, the length of which is at least 5. The cycle contains at least 5 vertices. For every vertex on the Co, it is adjacent to at least one edge which is not on the cycle and the adjacent vertices are distinct. (Otherwise, there exists a cycle whose length is less than 5.) Then it is easy to see 1 V I ;;: 2 X 5 = 10. Graph Theory 126 On the other hand, 2 1 E 1 = ~d (V) > ~3 = 3 v EV 4, so 2 1 V 1 + 8 ;;:, 3 1 VI, 1 V I. 1 E 1 = 1V 1 + v EV 1 V 1 ~ 8. A contradiction! Such counterexample does not exist. We have proven the proposition. Remark The method of proof in this problem is the proof by contradiction. To prove that the proposition is true, we assume that the proposition is false. Consider a variable V E N. From the smallest counterexample of V we can deduce a contradiction and the proof becomes easier using this condition. The conclusion of this problem is the best. When 1 E 1 = 1 V 1 + 3, we can give a counterexample as Fig. 5 shows us. 10 Fig. 5 We denote 21 persons by 21 vertices. There is an edge joining two vertices if and only if the two delegates which are represented by the two vertices have called each other. By assumption, there exists an odd cycle whose length is m. (We call cycle whose length is odd cycle.) Let C be the smallest odd cycle in the graph, the length of which is 2k + 1. 1 , let C be a triangle. It means the three persons have made a phone call to each other. Vj. that If k = If k > 1, set C and there is no edge joining Vi and +1, i - j *±1 (mod2k +1).) Otherwise, suppos (1 ~ i, j ~2k Vi' Vj Vl V2 • .• V2k+l Vl are adjacent and the sum of the length of cycle Vl V2 • •• V2k+l Vl and that of the cycle ViVi+l • . . VjVi is 2k + 3. So among them there must be an odd cycle whose length is less than 2k + 1. It V iV j ••• contradicts the fact that C is the shortest. Suppose there is no triangle among the 21 - (2k vertices other than at least (10 - k)2 Vl' V2' ••• , V2k+l. + 1) = 20 - 2k By Turan's theorem, there are edges joining them. Any vertex among them is not adjacent to two vertices which are adjacent to C, so it is adjacent to at most k vertices. So the sum of the edges is: Solutions = + 1 + H20 - 2k) + 00 100 + 2k + 1 - k = 102 - ~ 102 - (2 _ 1)2 2k 127 k)2 2 (k - 1) 2 = 101. A contradiction! So the graph must contain a triangle, which means there exist three persons who have called each other. 11 Suppose there exists such a graph that the degree of every vertex is more than 2. But the length of any cycle of the graph is divisible by 3. We consider the graph G which has this property and the least number of vertices. Clearly , the graph contains the shortest circle Z. The non-adjacent vertices on this cycle are not joined by an edge. Since the degree of every vertex is more than 2, every vertex on the cycle Z is adjacent to one vertex not on the cycle . Let Z pass the vertex A 1 , A 2 ' ••. , A 3k . Suppose that there exists a path 5 which joins the vertices A m and A " and which does not include edges in Z. We consider the cycle Z 1 and Z z consisting of the two halves of 5 and Z. Since the length of each of the two cycle is divisible by 3, it is not difficult to see the length of path 5 is divisible by 3. Especially, for the given graph , we can know that any vertex X which is not on the circle Z cannot have edges incident to two distinct vertices of Z. It means that the edges which are induced by the vertices on Z but not on the cycle should be incident to distinct vertices, respectively. Let us construct a graph G 1 • Collapse all vertices A 1 , A z , ... , A 3k on the cycle Z of G into one vertex A and keep all the vertices which are not on the cycle and their incident edges. Join the A and the vertices on the Z one by one. It is easy to know the degree of A is no less than 3k. The number of vertices in G 1 is less than that of G and the degree of every vertex is still more than 2. According to above conclusion , the length of any cycle in G is divisible by 3. We arrive at a contradiction . In view of the above proof, we can know G is the Graph Theory 128 graph satisfying these properties and with the least vertices. Then in a graph the degrees of its vertices are all more than 2, there must exist a cycle whose length can be divided by 3. Then we only need to apply this assertion to our problem. We denote the city by a vertex and the path by the edge. Exercise 5 1 When n en ~ 2) is odd, K" is a cycle. When n = 2, K 2 is a chain. When m, n are both even, K m ." is a cycle. 2 Suppose G contains at least 2k odd vertices. Delete one edge to get G'. There are three cases: e1) The number of odd vertices of G' decreases by 2, then G' need at least k - 1 strokes to draw. (2) The number of odd vertices increases by 2, G' need at least k + 1 strokes to draw. (3) The number of odd vertices does not change, G' need at least k strokes to draw. 3 These two graphs are all unicursal, that is, they can be drawn in one strock, and start and end at the same vertex. 4 When n is odd, the graph is unicursal; when n IS even, the graph is not unicursal. 5 Draw G as follows. We denote the persons by vertices. If two person have exchanged views, then join the corresponding vertices. Take the longest chain 1-'-' Let Vl be an end of 1-'-' then () vertices V2' ••• , VS+l adjacent to Vl are all on the chain 1-'-' Otherwise, I-'- can still be extended. Go along the I-'- through vertices V2' V3' •. • , VS+l' and then return to v 1 . This is a cycle whose length is more than (). v; 6 Take a point (j = 1, 2, 3, 4) on every face. If the two faces have common edges, join the two points. Then we get a new graph G * which we call the dual graph of G. In the graph G, going from a face to another face through the edge e i is equivalent to going from one vertex to another vertex along an edge in the G *. Therefore, if G v; Vj Fig. 6 Solutions 129 contains a broken line f1 satisfying conditions (1) and (2), then G ' is a chain (Q1 and Q2 are not on a face) or a cycle (Q1 and Q 2 are on a face), i. e. the graph G ' is unicursal or it can be drawn in one strock. But if the four vertices of G * are all odd, the graph G' needs two strocks to draw. 7 Suppose that there are k lines and that one vertex corresponds to one number ai = 1; Vi ai in the following way. If is blue, thenai =-1, i = 1,2, ... , n. V i is red, then Vi Then hence k is odd. 8 A ___ 1!" __ Series: Mathematical Olympiad SERIES 3 Year: 2,010 Pages: 156 sidor ; 22.9 cm Pages In File: 159 Language: English Identifier: 9789814271127,9814271128 Org File Size: 21,019,586 Extension: pdf

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